2050C Special Tutorial

We review 3.1 in Text.

1.1. Definition. A sequence $\{a_n\}$ is convergent to a if $\forall \varepsilon > 0$, $\exists n_\varepsilon$ s.t. $|a_n - a| < \varepsilon$, $\forall n \geq n_\varepsilon$. It is properly divergent to ∞ if $\forall M > 1$, $\exists n_M$ s.t. $a_n > M$, $\forall n \geq n_M$. It is called divergent if not convergent. Properly divergence is a special case of divergence.

e.g.1 $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ conv. to 0

$\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots\}$ conv. to 1

$\{1, 2, 3, 4, \cdots\}$ properly div.

$\{1, 0, 1, 0, \cdots\}$ divergent but not properly div.

Remarks (a) notations: $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} a_n = \infty$, $a_n \to a$, $a_n \to \infty$, etc are in use.

(b) n_ε depends on ε but is not unique.

(c) Any conv. sequence must be bounded, ie, $\exists M$ s.t. $|a_n| \leq M$, $\forall n$.

1.2 Basic Facts

- $\lim\limits_{n\to\infty} \frac{1}{n^p} = 0$ for any $p > 0$.
- $\lim\limits_{n\to\infty} n^p = \infty$ for any $p > 0$.

Write down the pfs yourself.

You may always quote these facts in exam without proving them.

1.3 Some Examples

e.g. 2 Find and show the limit of $\frac{2n+3}{n^2+1}$, $n \geq 1$.

First, as n becomes large $2n+3$ is dominated by $2n$ and n^2+1 by n^2. Hence the limit should be the same as $2n/n^2 = 2/n \to 0$.

Second, $\vdash \lim_{n\to\infty} \frac{2n+3}{n^2+1} = 0$.

$\left|\frac{2n+3}{n^2+1} - 0\right| < \varepsilon$, want to find n_ε such that this estimate holds for all $n \geq n_\varepsilon$. We have

$$\left|\frac{2n+3}{n^2+1} - 0\right| = \frac{2n+3}{n^2+1} \leq \frac{2n+3n}{n^2+1} < \frac{5n}{n^2} = \frac{5}{n},$$

So, $\left|\frac{2n+3}{n^2+1} - 0\right| < \varepsilon$ whenever $\frac{5}{n} < \varepsilon$. We choose $n_\varepsilon \geq [\frac{5}{\varepsilon}]+1$.

Then for $n \geq n_\varepsilon$, $n \geq [\frac{5}{\varepsilon}]+1 > \frac{5}{\varepsilon}$ ie, $\frac{5}{n} < \varepsilon$ and $\left|\frac{2n+3}{n^2+1} - 0\right| < \varepsilon$, done.

e.g. 3 Find and establish the limit of $\left\{\frac{3n^2+1}{n^2+n}\right\}$.

$3n^2+1 \sim 3n^2$, $n^2+n \sim n^2$, so guess $\frac{3n^2+1}{n^2+n} \to 3$

$$\left|\frac{3n^2+1}{n^2+n} - 3\right| = \frac{3n-1}{n^2+n} < \frac{3n}{n^2+n} < \frac{3n}{n^2} = \frac{3}{n} < \varepsilon.$$

For $\varepsilon > 0$, let $n_\varepsilon = [\frac{3}{\varepsilon}]+1$. then for $n \geq n_\varepsilon$, $\frac{3}{n} < \varepsilon$, so

$\left|\frac{3n^2+1}{n^2+n} - 3\right| < \varepsilon$, done.

1.4 More Basic Facts

- For $0<a<1$, $\lim_{n\to\infty} a^n = 0$.
- For $b>1$, $\lim_{n\to\infty} b^n = \infty$.
- For $a>0$, $\lim_{n\to\infty} \sqrt[n]{a} = 1$.
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

The first three facts are based on Bernoulli's inequality

$$(1+x)^n \geqslant 1+nx, \quad \forall n \geqslant 1,\ x>-1 \qquad \text{(Pg 30 Text)}$$

The last one is based on Binomial theorem.

You MUST understand the pfs well.

You may quote them directly unless you are asked to prove them.

1.5 Exercises

Ex 1 Find and establish the limits:

(a) $\lim_{n\to\infty} \frac{n^2-1}{n^3+6}$, (b) $\lim_{n\to\infty} \frac{n+5}{n-2}$,

(c) $\lim_{n\to\infty} \sqrt{n^2+1} - n$, (d) $\lim_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$.

Ex 2. Prove $\lim_{n\to\infty} a^n = 0$ for $a \in (0,1)$.

Ex 3. Prove $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for $a>0$.